

BOUNDEDNESS OF MODULI OF VARIETIES OF GENERAL TYPE

CHRISTOPHER D. HACON, JAMES M^CKERNAN, AND CHENYANG XU

ABSTRACT. We show that the family of semi log canonical pairs
with ample log canonical class and with fixed volume is bounded.

CONTENTS

1. Introduction	2
2. Preliminaries	7
2.1. Notations and Conventions	7
2.2. The volume	8
2.3. Deformation Invariance	9
2.4. DCC sets	11
2.5. Semi log canonical varieties	12
2.6. Base Point Free Theorem	13
2.7. Minimal models	13
2.8. Blowing up log pairs	14
2.9. Good minimal models	16
3. The MMP in families I	19
4. Invariance of plurigenera	21
5. The MMP in families II	23
6. Abundance in families	26
7. Boundedness of moduli	32
References	36

Date: December 4, 2014.

The first author was partially supported by DMS-1300750, DMS-1265285 and a grant from the Simons foundation, the second author was partially supported by NSF research grant no: 0701101, no: 1200656 and no: 1265263 and this research was partially funded by the Simons foundation and by the Mathematisches Forschungsinstitut Oberwolfach and the third author was partially supported by “The Recruitment Program of Global Experts” grant from China. Part of this work was completed whilst the second and third authors were visiting the Freiburg Institute of Advanced Studies and they would like to thank Stefan Kebekus and the Institute for providing such a congenial place to work. We are grateful to János Kollár and Mihai Păun for many useful comments and suggestions.

1. INTRODUCTION

The aim of this paper is to show that the moduli functor of semi log canonical stable pairs is bounded:

Theorem 1.1. *Fix an integer n , a positive rational number d and a set $I \subset [0, 1]$ which satisfies the DCC.*

Then the set $\mathfrak{F}_{slc}(n, d, I)$ of all log pairs (X, Δ) such that

- (1) X is projective of dimension n ,*
- (2) (X, Δ) is semi log canonical,*
- (3) the coefficients of Δ belong to I ,*
- (4) $K_X + \Delta$ is an ample \mathbb{Q} -divisor, and*
- (5) $(K_X + \Delta)^n = d$,*

is bounded.

In particular there is a finite set I_0 such that $\mathfrak{F}_{slc}(n, d, I) = \mathfrak{F}_{slc}(n, d, I_0)$.

The main new technical result we need to prove (1.1) is to show that abundance behaves well in families:

Theorem 1.2. *Suppose that (X, Δ) is a log pair where the coefficients of Δ belong to $(0, 1] \cap \mathbb{Q}$. Let $\pi: X \rightarrow U$ be a projective morphism to a smooth variety U . Suppose that (X, Δ) is log smooth over U .*

If there is a closed point $0 \in U$ such that the fibre (X_0, Δ_0) has a good minimal model then (X, Δ) has a good minimal model over U and every fibre has a good minimal model.

Corollary 1.3. *Let (X, Δ) be a log pair where Δ is a \mathbb{Q} -divisor and let $X \rightarrow U$ be a projective morphism to a variety U .*

Then the subset $U_0 \subset U$ of points $u \in U$ such that the fibre (X_u, Δ_u) is divisorially log terminal and has a good minimal model is constructible.

Corollary 1.4. *Let $\pi: X \rightarrow U$ be a projective morphism to a smooth variety U and let (X, Δ) be log smooth over U . Suppose that the coefficients of Δ belong to $(0, 1] \cap \mathbb{Q}$.*

If there is a closed point $0 \in U$ such that the fibre (X_0, Δ_0) has a good minimal model then the restriction morphism

$$\pi_* \mathcal{O}_X(m(K_X + \Delta)) \rightarrow H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u)))$$

is surjective for any $m \in \mathbb{N}$ such that $m\Delta$ is integral and for any closed point $u \in U$.

In particular if $\psi: X \dashrightarrow Z$ is the ample model of (X, Δ) then $\psi_u: X_u \dashrightarrow Z_u$ is the ample model of (X_u, Δ_u) for every closed point $u \in U$.

The moduli space of stable curves is one of the most intensively studied varieties. The moduli space of stable varieties of general type

is the higher dimensional analogue of the moduli space of curves. Unfortunately constructing this moduli space is more complicated than constructing the moduli space of curves. In particular it does not seem easy to use GIT to construct the moduli space in higher dimensions; for example see [28] for a precise example of how badly behaved the situation can be. Instead Kollár and Shepherd-Barron initiated a program to construct the moduli space in all dimensions in [25]. This program was carried out in large part by Alexeev for surfaces, [1] and [2].

We recall the definition of the moduli functor. For simplicity, in the definition of the functor, we restrict ourselves to the case with no boundary. We refer to the forthcoming book [17] for a detailed discussion of this subject and to [22] for a more concise survey.

Definition 1.5 (Moduli of slc models, cf. [22, 29]). *Let $H(m)$ be an integer valued function. The moduli functor of semi log canonical models with Hilbert function H is*

$$\mathcal{M}_H^{slc}(S) = \left\{ \begin{array}{l} \text{flat projective morphisms } X \longrightarrow S, \text{ whose} \\ \text{fibres are slc models with ample canonical class} \\ \text{and Hilbert function } H(m), \omega_X \text{ is flat over } S \\ \text{and commutes with base change.} \end{array} \right\}$$

In this paper we focus on the problem of showing that the moduli functor is bounded, so that if we fix the degree, we get a bounded family. The precise statement is given in (1.1). We now describe the proof of (1.1). We first explain how to reduce to (1.2).

For curves if one fixes the genus g then the moduli space is irreducible. In particular stable curves are always limits of smooth curves. This fails in higher dimensions, so that there are components of the moduli space whose general point corresponds to a non-normal variety, or better, a semi log canonical variety.

Fortunately, cf. [21, 23, 24] and [23, 5.13], one can reduce boundedness of semi log canonical pairs to boundedness of log canonical pairs in a straightforward manner. If (X, Δ) is semi log canonical then let $n: Y \longrightarrow X$ be the normalisation. X has nodal singularities in codimension one, so that informally X is obtained from Y by identifying points of the double locus, the closure of the codimension one singular locus. More precisely, we may write

$$K_Y + \Gamma = n^*(K_X + \Delta),$$

where Γ is the sum of the strict transform of Δ plus the double locus and (Y, Γ) is log canonical. If $K_X + \Delta$ is ample then (X, Δ) is determined by (Y, Γ) and the data of the involution $\tau: S \longrightarrow S$ of the normalisation

of the double locus. Note that the involution τ fixes the different, the divisor Θ defined by adjunction in the following formula:

$$(K_Y + \Gamma)|_S = K_S + \Theta.$$

Conversely, if (Y, Γ) is log canonical, $K_Y + \Gamma$ is ample, τ is an involution of the normalisation S of a divisor supported on $[\Gamma]$ which fixes the different, then we may construct a semi log canonical pair (X, Δ) , whose normalisation is (Y, Γ) and whose double locus is S .

Note that τ fixes the pullback L of the very ample line bundle determined by a multiple of $K_X + \Delta$. The group of all automorphisms of S which fixes L is a linear algebraic group. It follows, by standard properties of the scheme Isom , that if (Y, Γ) is bounded then τ is bounded.

Thus to prove (1.1) it suffices to prove the result, when X is normal, that is, when (X, Δ) is log canonical, cf. (7.3). The first problem is that a priori X might have arbitrarily many components. Note that if $X = C$ is a curve of genus g then K_X has degree $2g - 2$ and so X has at most $2g - 2$ components. In higher dimensions the situation is more complicated since K_X is not necessarily Cartier and so d is not necessarily an integer.

Instead we use [12, 1.3], which was conjectured by Alexeev [1] and Kollár [19]:

Theorem 1.6. *Fix a positive integer n and a set $I \subset [0, 1]$ which satisfies the DCC. Let \mathfrak{D} be the set of log canonical pairs (X, Δ) such that the dimension of X is n and the coefficients of Δ belong to I .*

Then the set

$$\{ \text{vol}(X, K_X + \Delta) \mid (X, \Delta) \in \mathfrak{D} \},$$

also satisfies the DCC.

Since there are only finitely many ways to write d as a sum of elements d_1, d_2, \dots, d_k taken from a set which satisfies the DCC, cf. (2.4.1), we are reduced to proving (1.1) when X is normal and irreducible.

Let $\mathfrak{F} \subset \mathfrak{F}_{\text{slc}}(n, d, I)$ be the subset of all log canonical pairs (X, Δ) where X is irreducible. Since the coefficients of Δ belong to a set which satisfies the DCC, [12, 1.3] implies that some fixed multiple of $K_X + \Delta$ defines a birational map to projective space. As the degree of $K_X + \Delta$ is bounded by assumption, \mathfrak{F} is log birationally bounded, that is, there is a log pair (Z, B) and a projective morphism $\pi: Z \rightarrow U$, such that given any $(X, \Delta) \in \mathfrak{F}$, we may find $u \in U$ such that X is birational to

Z_u and the strict transform Φ of Δ plus the exceptional divisors are components of B_u .

[12, 1.6] proves that \mathfrak{F} is a bounded family provided if in addition we assume that the total log discrepancy of (X, Δ) is bounded away from zero (meaning that the coefficients of Δ are bounded away from one as well as the log discrepancy is bounded away from zero). For applications to moduli this is far too strong; the double locus occurs with coefficient one.

Instead we proceed as follows. By standard arguments we may assume that U , is smooth the morphism π is smooth and its restriction to any strata of B is smooth, that is, (Z, B) is log smooth over U . We first reduce to the case when $\text{vol}(Z_u, K_{Z_u} + \Phi) = d$. We are looking for a higher model $Y \rightarrow Z$ such that $\text{vol}(Y, K_Y + \Gamma) = d$ where Γ is the transform of Δ plus the exceptionals. At this point we use some of the ideas that go into the proof of [11, 1.9]. By deformation invariance of log plurigenera we may assume that U is a point, (7.2).

In general $\text{vol}(Z_u, K_{Z_u} + \Phi) \geq \text{vol}(X, K_X + \Delta) = d$. Since the volume satisfies the DCC, (1.6), we may assume that the model Z minimises the supremum of $\text{vol}(Z, K_Z + \Phi)$. In this case, by a standard diagonalisation argument, we are given a sequence of pairs $(X_i, \Delta_i) \in \mathfrak{F}$ and it suffices to find a higher model $Y \rightarrow Z$ where the limit has smaller volume. This follows using some results from [11], cf. (7.1).

So we may assume that $\text{vol}(Z_u, K_{Z_u} + \Phi) = d$. Since (X, Δ) is log canonical and $K_X + \Delta$ is ample, we can recover (X, Δ) from (Z_u, Φ) as the log canonical model, cf. (2.2.2). Conversely if $u \in U$ is a point such that (Z_u, Φ) has a log canonical model, $f: Z_u \dashrightarrow X$, where

$$X = \text{Proj } R(Z_u, K_{Z_u} + \Phi) \quad \text{and} \quad \Delta = f_* \Phi,$$

the coefficients of $0 \leq \Phi \leq B_u$ belong to I and $\text{vol}(Z_u, K_{Z_u} + \Phi) = d$ then $(X, \Delta) \in \mathfrak{F}$.

It therefore suffices to prove that the set of fibres with a log canonical model is constructible. Note that (X, Δ) has a log canonical model if and only if the log canonical section ring

$$R(X, K_X + \Delta) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$$

is finitely generated. Conjecturally every fibre has a log canonical model. Once again the problem are the components of Δ with coefficient one. The main result of [6] implies that if there are no components of Δ with coefficient one, that is, (X, Δ) is kawamata log terminal, then the log canonical section ring is finitely generated.

In general, (2.9.1), the existence of the log canonical model Z is equivalent to the existence of a good minimal model $f: X \dashrightarrow Y$, that is, a model (Y, Γ) such that $K_Y + \Gamma$ is semi-ample. In this case the log canonical model is simply the model $Y \rightarrow Z$ such that $K_Y + \Gamma$ is the pullback of an ample divisor.

In fact we prove, (1.2), a much stronger result. We prove that if one fibre (X_0, Δ_0) has a good minimal model then every fibre has a good minimal model. By [14, 1.1] it suffices to prove that every fibre over an open subset has a good minimal model, equivalently, that the generic fibre has a good minimal model.

Let $\eta \in U$ be the generic point. We may assume that U is affine. We prove the existence of a good minimal model for the pair (X_η, Δ_η) in two steps. We first show that (X_η, Δ_η) has a minimal model. For this we run the $(K_X + \Delta)$ -MMP with scaling of an ample divisor. We know that if we run the $(K_{X_0} + \Delta_0)$ -MMP with scaling of an ample divisor then this MMP terminates with a good minimal model. However, if we run the $(K_X + \Delta)$ -MMP we might lose the property that X_0 is irreducible, the flipping locus might be an extra component of the new central fibre. Using [14, 2.10] and (5.3) we can reduce to the case when the diminished stable base locus of $K_{X_0} + \Delta_0$ does not contain any non canonical centres. In this case we show, (3.1), that every step of the $(K_X + \Delta)$ -MMP induces a $(K_{X_0} + \Delta_0)$ -negative map. This generalises [11, 4.1], which assumes that U is a curve and that (X, Δ) is terminal. This MMP ends $f: X \dashrightarrow Y$ with a minimal model for the generic fibre, (3.2).

To finish off we need to show that the minimal model is a good minimal model. There are two cases. We may write $(X, \Delta = S + B)$, where $S = \lfloor \Delta \rfloor$.

In the first case, if $K_X + (1 - \epsilon)S + B$ is not pseudo-effective for any $\epsilon > 0$ then we may run $Y \dashrightarrow W$ the $(K_X + (1 - \epsilon)S + B)$ -MMP until we reach a Mori fibre space (5.2) $W \rightarrow Z$. If $\epsilon > 0$ is sufficiently small, this MMP induces a $(K_{X_0} + \Delta_0)$ -non-positive map, see (5.1). It follows that this MMP is $(K_X + \Delta)$ -non-positive. We know that there is a component D of S whose image dominates the base Z of the Mori fibre space. By induction the generic fibre of the image E of D in Y is a good minimal model. The restriction $E \dashrightarrow F$ of the map $Y \dashrightarrow W$ need not be a birational contraction but we won't lose semi-ampleness. The image of the divisor is pulled back from Z and so $K_X + \Delta$ has a semi-ample model.

In the second case $K_X + (1 - \epsilon)S + B$ is pseudo-effective. As $K_X + (1 - \epsilon)S + B$ is kawamata log terminal, it follows by work of B. Berndtsson and M. Păun, (4.1), that the Kodaira dimension is invariant, see

(4.2). As $K_X + (1 - \epsilon)S + B$ is pseudo-effective and (X_0, Δ_0) has a good minimal model, it follows that $K_{X_0} + \Delta_0$ is abundant, that is, the Kodaira dimension is the same as the numerical dimension. By deformation invariance of log plurigenera the generic fibre is abundant. As the restriction of $K_Y + \Gamma$ to every component of coefficient one is semi-ample, the restriction of $K_Y + \Gamma$ to the sum of the coefficient one part is semi-ample by (2.5.1) and we are done by (2.6.1).

2. PRELIMINARIES

2.1. Notations and Conventions. We will follow the terminology from [24]. Let $f: X \dashrightarrow Y$ be a proper birational map of normal quasi-projective varieties and let $p: W \rightarrow X$ and $q: W \rightarrow Y$ be a common resolution of f . We say that f is a *birational contraction* if every p -exceptional divisor is q -exceptional. If D is an \mathbb{R} -Cartier divisor on X such that $D' := f_*D$ is \mathbb{R} -Cartier then we say that f is *D -non-positive* (resp. *D -negative*) if we have $p^*D = q^*D' + E$ where $E \geq 0$ and E is q -exceptional (respectively E is q -exceptional and the support of E contains the strict transform of the f -exceptional divisors).

We say a proper morphism $\pi: X \rightarrow U$ is a *contraction morphism* if $\pi_*\mathcal{O}_X = \mathcal{O}_U$. Recall that for any divisor D on X , the sheaf $\pi_*\mathcal{O}_X(D)$ is defined to be $\pi_*\mathcal{O}_X(\lfloor D \rfloor)$. If we are given a morphism $X \rightarrow U$, then we say that (X, Δ) is *log smooth over U* if (X, Δ) has simple normal crossings and both X and every stratum of (X, D) is smooth over U , where D is the support of Δ . If $\pi: X \rightarrow U$ and $Y \rightarrow U$ are projective morphisms, $f: X \dashrightarrow Y$ is a birational contraction over U and (X, Δ) is a log canonical pair (respectively divisorially log terminal \mathbb{Q} -factorial pair) such that f is $(K_X + \Delta)$ -non-positive (respectively $(K_X + \Delta)$ -negative) and $K_Y + \Gamma$ is nef over U (respectively and Y is \mathbb{Q} -factorial), then we say that $f: X \dashrightarrow Y$ is a *weak log canonical model* (respectively *a minimal model*) of $K_X + \Delta$ over U .

We say $K_Y + \Gamma$ is *semi-ample* over U if there exists a surjective morphism $\psi: Y \rightarrow Z$ over U such that $K_Y + \Gamma \sim_{\mathbb{R}} \psi^*A$ for some \mathbb{R} -divisor A on Y which is ample over U . Equivalently, when $K_Y + \Gamma$ is \mathbb{Q} -Cartier, $K_Y + \Gamma$ is semi-ample over U if there exists an integer $m > 0$ such that $\mathcal{O}_Y(m(K_Y + \Gamma))$ is generated over U . Note that in this case

$$R(Y/U, K_Y + \Gamma) := \bigoplus_{m \geq 0} \pi_*\mathcal{O}_Y(m(K_Y + \Gamma))$$

is a finitely generated \mathcal{O}_U -algebra, and

$$Z = \text{Proj } R(Y/U, K_Y + \Gamma).$$

If $K_Y + \Gamma$ is semi-ample and big over U , then Z is the *log canonical model* of (X, Δ) over U . A weak log canonical model $f: X \dashrightarrow Y$ is called a *semi-ample model* if $K_Y + \Gamma$ is semi-ample.

Let D be an \mathbb{R} -Cartier divisor on a projective variety X . Let C be a prime divisor. If D is big then

$$\sigma_C(D) = \inf\{\text{mult}_C(D') \mid D' \sim_{\mathbb{R}} D, D' \geq 0\}.$$

Now let A be any ample \mathbb{Q} -divisor. Following [27], let

$$\sigma_C(D) = \lim_{\epsilon \rightarrow 0} \sigma_C(D + \epsilon A).$$

Then $\sigma_C(D)$ exists and is independent of the choice of A . There are only finitely many prime divisors C such that $\sigma_C(D) > 0$ and the \mathbb{R} -divisor $N_\sigma(X, D) = \sum_C \sigma_C(D)C$ is determined by the numerical equivalence class of D , cf. [6, 3.3.1] and [27] for more details.

Following [27] we define *the numerical dimension*

$$\kappa_\sigma(X, D) = \max_{H \in \text{Pic}(X)} \{k \in \mathbb{N} \mid \limsup_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(mD + H))}{m^k} > 0\}.$$

If D is nef then this is the same as

$$\nu(X, D) = \max\{k \in \mathbb{N} \mid H^{n-k} \cdot D^k > 0\}$$

for any ample divisor H , see [27]. D is called *abundant* if $\kappa_\sigma(X, D) = \kappa(X, D)$, that is, the numerical dimension is equal to the Iitaka dimension. If we drop the condition that X is projective and instead we have a projective morphism $\pi: X \rightarrow U$, then an \mathbb{R} -Cartier divisor D on X , is called *abundant* over U if its restriction to the generic fibre is abundant.

If (X, Δ) is a log pair then a *non canonical centre* is the centre of a valuation of log discrepancy less than one.

2.2. The volume.

Definition 2.2.1. Let X be a normal n -dimensional irreducible projective variety and let D be an \mathbb{R} -divisor. The **volume** of D is

$$\text{vol}(X, D) = \limsup_{m \rightarrow \infty} \frac{n! h^0(X, \mathcal{O}_X(mD))}{m^n}.$$

Let $V \subset X$ be an irreducible subvariety of dimension d . Suppose that D is \mathbb{R} -Cartier. The **restricted volume** of D along V is

$$\text{vol}(X|V, D) = \limsup_{m \rightarrow \infty} \frac{d! (\dim \text{Im}(H^0(X, \mathcal{O}_X(mD)) \rightarrow H^0(V, \mathcal{O}_V(mD))))}{m^d}.$$

Lemma 2.2.2. *Let $f: X \rightarrow Z$ be a birational morphism between log canonical pairs (X, Δ) and (Z, B) . Suppose that $K_X + \Delta$ is big and that (X, Δ) has a log canonical model $g: X \dashrightarrow Y$.*

If $f_\Delta \leq B$ and $\text{vol}(X, K_X + \Delta) = \text{vol}(Z, K_Z + B)$ then the induced birational map $Z \dashrightarrow Y$ is the log canonical model of (Z, B) .*

Proof. Let $\pi: W \rightarrow X$ be a log resolution of $(X, C + F)$, which also resolves the map g , where C is the strict transform of B and F is the sum of the f -exceptional divisors. We may write

$$K_W + \Theta = \pi^*(K_X + \Delta) + E,$$

where $\Theta \geq 0$ and $E \geq 0$ have no common components, $\pi_*\Theta = \Delta$ and $\pi_*E = 0$. Then the log canonical model of (W, Θ) is the same as the log canonical model of (X, Δ) . Replacing (X, Δ) by (W, Θ) we may assume that $(X, C + F)$ is log smooth and $g: X \rightarrow Y$ is a morphism. Replacing (Z, B) by the pair $(X, D = C + F)$, we may assume $Z = X$.

If $A = g_*(K_X + \Delta)$ and $H = g^*A$ then A is ample and $K_X + \Delta - H \geq 0$. Let $L = D - \Delta \geq 0$, let S be a component of L with coefficient a and let

$$v(t) = \text{vol}(X, H + tS).$$

Then $v(t)$ is a non-decreasing function of t and

$$\begin{aligned} v(0) &= \text{vol}(X, H) \\ &= \text{vol}(X, K_X + \Delta) \\ &= \text{vol}(X, K_X + D) \\ &\geq \text{vol}(X, H + L) \\ &\geq \text{vol}(X, H + aS) \\ &= v(a). \end{aligned}$$

Thus $v(t)$ is constant over the range $[0, a]$. [26, 4.25 (iii)] implies that

$$\left. \frac{1}{n} \frac{dv}{dt} \right|_{t=0} = \text{vol}_{X|S}(H) \geq S \cdot H^{n-1} = g_*S \cdot A^{n-1}$$

so that $g_*S = 0$. But then every component of L is exceptional for g and g is the log canonical model of (X, D) . \square

2.3. Deformation Invariance.

Lemma 2.3.1. *Let $\pi: X \rightarrow U$ be a projective morphism to a smooth variety U and let (X, Δ) be a log smooth pair over U .*

If the coefficients of Δ belong to $[0, 1]$ then

$$N_\sigma(X, K_X + \Delta)|_{X_u} = N_\sigma(X_u, K_{X_u} + \Delta_u)$$

for every $u \in U$.

Proof. Pick a relatively ample Cartier divisor A such that $(X, \Delta + A)$ is log smooth over U . Fix $u \in U$. Then [11, 1.8.1] implies that

$$f_* \mathcal{O}_X(m(K_X + \Delta) + A) \longrightarrow H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u) + A_u))$$

is surjective for all positive integers m such that $m\Delta$ is integral. It follows that

$$N_\sigma(X, K_X + \Delta)|_{X_u} \leq N_\sigma(X_u, K_{X_u} + \Delta_u)$$

and the reverse inequality is clear. \square

Lemma 2.3.2. *Let $\pi: X \longrightarrow U$ be a projective morphism to a smooth variety U and let (X, Δ) be a log smooth pair over U . Let $0 \in U$ be a closed point, let*

$$\Theta_0 = \Delta_0 - \Delta_0 \wedge N_\sigma(X_0, K_{X_0} + \Delta_0)$$

and let $0 \leq \Theta \leq \Delta$ be the unique divisor so that $\Theta_0 = \Theta|_{X_0}$.

If the coefficients of Δ belong to $[0, 1]$ then

$$\Theta = \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta).$$

Proof. Fix a positive integer k such that $k\Delta \geq \lceil \Delta \rceil$. Pick a relatively ample Cartier divisor H such that $(X, \Delta + H)$ is log smooth over U and $kK_X + H$ is big. Let $m > k$ be an integer. Consider the commutative diagram

$$\begin{array}{ccc} \pi_* \mathcal{O}_X(m(K_X + \Theta) + H) & \longrightarrow & \pi_* \mathcal{O}_X(m(K_X + \Delta) + H) \\ \downarrow & & \downarrow \\ H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Theta_0) + H_0)) & \longrightarrow & H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0) + H_0)). \end{array}$$

The top row is an inclusion and the bottom row is an isomorphism by assumption. As

$$\lfloor m(K_X + \Delta) \rfloor + H = (m - k)(K_X + \Delta) + kK_X + H + (k\Delta - \{m\Delta\})$$

is big, the first column is surjective by [11, 1.8.1]. Nakayama's Lemma implies that the top row is an isomorphism in a neighbourhood of X_0 . It follows that

$$\Theta \geq \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta).$$

and the reverse inequality follows by (2.3.1). \square

Lemma 2.3.3. *Let $\pi: X \longrightarrow U$ be a projective morphism to a smooth variety U and let (X, D) be log smooth over U , where the coefficients of D are all one. Let $0 \in U$ be a closed point.*

Then the restriction morphism

$$\pi_* \mathcal{O}_X(K_X + D) \longrightarrow H^0(X_0, \mathcal{O}_{X_0}(K_{X_0} + D_0))$$

is surjective.

Proof. Since the result is local we may assume that U is affine. Cutting by hyperplanes we may assume that U is a curve. Thus we want to show that the restriction map

$$H^0(X, \mathcal{O}_X(K_X + X_0 + D)) \longrightarrow H^0(X_0, \mathcal{O}_{X_0}(K_{X_0} + D_0))$$

is surjective. This is equivalent to showing that multiplication by a local parameter

$$H^1(X, \mathcal{O}_X(K_X + D)) \longrightarrow H^1(X, \mathcal{O}_X(K_X + D + X_0))$$

is injective.

By assumption the image of every strata of D is the whole of U and $0 = (K_X + D) - (K_X + D)$ is semi-ample. Therefore a generalisation of Kollár's injectivity theorem (see [18], [8, 6.3] and [4, 5.4]) implies that

$$H^1(X, \mathcal{O}_X(K_X + D)) \longrightarrow H^1(X, \mathcal{O}_X(K_X + D + X_0))$$

is injective. \square

2.4. DCC sets.

Lemma 2.4.1. *Let $I \subset \mathbb{R}$ be a set of positive real numbers which satisfies the DCC. Fix a constant d .*

Then the set

$$T = \{ (d_1, d_2, \dots, d_k) \mid k \in \mathbb{N}, d_i \in I, \sum d_i = d \}$$

is finite.

Proof. As I satisfies the DCC there is a real number $\delta > 0$ such that if $i \in I$ then $i \geq \delta$. Thus

$$k \leq \frac{d}{\delta}.$$

It is enough to show that given any infinite sequence t_1, t_2, \dots of elements of T that we may find a constant subsequence. Possibly passing to a subsequence we may assume that the number of entries k of each vector $t_i = (d_{i1}, d_{i2}, \dots, d_{ik})$ is constant. Since I satisfies the DCC, possibly passing to a subsequence, we may assume that the entries are not decreasing. Since the sum is constant, it is clear that the entries are constant, so that t_1, t_2, \dots is a constant sequence. \square

Lemma 2.4.2. *Let J be a finite set of real numbers at most one.*

If

$$I = \{ a \in (0, 1] \mid a = 1 + \sum_{i \leq k} a_i - k, a_1, a_2, \dots, a_k \in J \}.$$

then I is finite.

Proof. If $a_k = 1$ then

$$\sum_{i \leq k} a_i - k = \sum_{i \leq k-1} a_i - (k-1).$$

Thus there is no harm in assuming that $1 \notin J$. If $a_k < 0$ then

$$1 + \sum_{i \leq k} a_i - k < 0.$$

Thus we may assume that $J \subset [0, 1)$.

Note that

$$1 + \sum_{i \leq k} a_i - k > 0 \quad \text{if and only if} \quad \sum_{i \leq k} (1 - a_i) < 1.$$

Since J is finite we may find $\delta > 0$ such that if $a \in J$ then $1 - a \geq \delta$. This bounds k and the result is clear. \square

2.5. Semi log canonical varieties. We will need the definition of certain singularities of semi-normal pairs, [20, 7.2.1]. Let X be a semi-normal variety which satisfies Serre's condition S_2 and let Δ be an \mathbb{R} -divisor on X , such that $K_X + \Delta$ is \mathbb{R} -Cartier. Let $n: Y \rightarrow X$ be the normalisation of X and write

$$K_Y + \Gamma = n^*(K_X + \Delta),$$

where Γ is the sum of the strict transform of Δ and the double locus. We say that (X, Δ) is *semi log canonical* if (Y, Γ) is log canonical and (X, Δ) is *divisorially semi log terminal* if (Y, Γ) is divisorially log terminal. Note that if (X, Δ) is divisorially log terminal and S is the union of the components of $\lfloor \Delta \rfloor$, then (S, Θ) is divisorially semi log terminal where

$$(K_X + \Delta)|_S = K_S + \Theta.$$

Theorem 2.5.1. *Let (X, Δ) be a semi log canonical pair and let $n: Y \rightarrow X$ be the normalisation. By adjunction we may write*

$$K_Y + \Gamma = n^*(K_X + \Delta),$$

where (Y, Γ) is log canonical.

If X is projective and Δ is a \mathbb{Q} -divisor then $K_X + \Delta$ is semi-ample if and only if $K_Y + \Gamma$ is semi-ample.

Proof. See [9] or [13, 1.4]. \square

Suppose that (X, Δ) is log canonical and $\pi: X \rightarrow U$ is a morphism of quasi-projective varieties. If (X_0, Δ_0) is the fibre over a closed point $0 \in U$ then note that

$$(K_X + \Delta)|_{X_0} = K_{X_0} + \Delta_0.$$

2.6. Base Point Free Theorem. Recall the following generalization of Kawamata's theorem:

Theorem 2.6.1. *Let $(X, \Delta = S+B)$ be a divisorially log terminal pair, where $S = \lfloor \Delta \rfloor$ and B is a \mathbb{Q} -divisor. Let H be a \mathbb{Q} -Cartier divisor on X and let $X \rightarrow U$ be a proper surjective morphism of varieties.*

If there is a constant a_0 such that

- (1) $H|_S$ is semi-ample over U ,*
- (2) $aH - (K_X + \Delta)$ is nef and abundant over U , for all $a > a_0$,*

then H is semi-ample over U .

Proof. See [15], [3], [8], [7], [9] and [14, 4.1]. □

2.7. Minimal models.

Lemma 2.7.1. *Let (X, Δ) be a divisorially log terminal pair where X is \mathbb{Q} -factorial and projective. Assume that $K_X + \Delta$ is pseudo-effective. Suppose that we run $f: X \dashrightarrow Y$ the $(K_X + \Delta)$ -MMP with scaling of an ample divisor A , so that $(Y, \Gamma + tB)$ is nef, where $\Gamma = f_*\Delta$ and $B = f_*A$.*

- (1) If F is f -exceptional then F is a component of $N_\sigma(X, K_X + \Delta)$.*
- (2) If $t > 0$ is sufficiently small then every component of $N_\sigma(X, K_X + \Delta)$ is f -exceptional.*
- (3) If (X, Δ) has a minimal model and $K_X + \Delta$ is \mathbb{Q} -Cartier then $N_\sigma(X, K_X + \Delta)$ is a \mathbb{Q} -divisor.*

Proof. Let $p: W \dashrightarrow X$ and $q: W \dashrightarrow Y$ resolve f . As f is a minimal model of $(X, tA + \Delta)$, for some $t \geq 0$, we may write

$$p^*(K_X + tA + \Delta) = q^*(K_Y + tB + \Gamma) + E,$$

where $E = E_t \geq 0$ is q -exceptional. As $q^*(K_Y + tB + \Gamma)$ is nef, it follows that

$$N_\sigma(X, K_X + tA + \Delta) = p_*E.$$

As A is ample, (1) holds. If t is sufficiently small then

$$N_\sigma(X, K_X + tA + \Delta) \quad \text{and} \quad N_\sigma(X, K_X + \Delta)$$

have the same support and so (2) holds.

If (X, Δ) has a minimal model then we may assume that $t = 0$ and so

$$N_\sigma(X, K_X + \Delta) = p_*E_0$$

is a \mathbb{Q} -divisor. □

Lemma 2.7.2. *Let (X, Δ) be a divisorially log terminal pair where X is \mathbb{Q} -factorial and projective. Assume that $K_X + \Delta$ is pseudo-effective.*

If $f: X \dashrightarrow Y$ is a birational contraction such that $K_Y + \Gamma = f_(K_X + \Delta)$ is nef and f only contracts components of $N_\sigma(X, K_X + \Delta)$ then f is a minimal model of (X, Δ) .*

Proof. Let $p: W \rightarrow X$ and $q: W \rightarrow Y$ resolve f . We may write

$$p^*(K_X + \Delta) + E = q^*(K_Y + \Gamma) + F,$$

where $E \geq 0$ and $F \geq 0$ have no common components and both E and F are q -exceptional.

As $K_Y + \Gamma$ is nef, the support of F and the support of $N_\sigma(Y, q^*(K_Y + \Gamma) + F)$ coincide. On the other hand, every component of E is a component of $N_\sigma(X, p^*(K_X + \Delta) + E)$. Thus $E = 0$ and any divisor contracted by f is a component of F . \square

2.8. Blowing up log pairs.

Lemma 2.8.1. *Let (X, Δ) be a log smooth pair.*

If $\lfloor \Delta \rfloor = 0$ then there is a sequence $\pi: Y \rightarrow X$ of smooth blow ups of the strata of (X, Δ) such that if we write

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $\pi_\Gamma = \Delta$ and $\pi_*E = 0$, then no two components of Γ intersect.*

Proof. This is standard, see for example [10, 6.5]. \square

Lemma 2.8.2. *Let (X, Δ) be a sub log canonical pair.*

We may find a finite set $I \subset (0, 1]$ such that if $\pi: Y \rightarrow X$ is any birational morphism and we write

$$K_Y + \Gamma = \pi^*(K_X + \Delta)$$

then the coefficients of Γ which are positive belong to I .

Proof. Replacing (X, Δ) by a log resolution we may assume that (X, Δ) is log smooth. Let J be the set of coefficients of Δ and let I be the set given by (2.4.2).

Suppose that $\pi: Y \rightarrow X$ is a birational morphism. We may write

$$K_Y + \Gamma = \pi^*(K_X + \Delta).$$

We claim that the coefficients of Γ which are positive belong to I . Possibly blowing up more we may assume that π is a sequence of smooth blow ups. If $Z \subset X$ is smooth of codimension k and a_1, a_2, \dots, a_k are

the coefficients of the components of Δ containing Z then the coefficient of the exceptional divisor is

$$a = 1 + \sum_{i \leq k} a_i - k.$$

If $a > 0$ then $a \in I$ and we are done by induction on the number of blow ups. \square

Lemma 2.8.3. *Let (X, Δ) be a log smooth pair where the coefficients of Δ belong to $(0, 1]$ and X is projective.*

If (X, Δ) has a weak log canonical model then there is a sequence $\pi: Y \rightarrow X$ of smooth blow ups of the strata of Δ such that if we write

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $\pi_\Gamma = \Delta$ and $\pi_*E = 0$ and if we write*

$$\Gamma' = \Gamma - \Gamma \wedge N_\sigma(Y, K_Y + \Gamma),$$

then $\mathbf{B}_-(Y, K_Y + \Gamma')$ contains no strata of Γ' . If Δ is a \mathbb{Q} -divisor then Γ' is a \mathbb{Q} -divisor.

Proof. Let $f: X \dashrightarrow W$ be a weak log canonical model of (X, Δ) . Let $\Phi = f_*\Delta$. Let I be the finite set whose existence is guaranteed by (2.8.2) applied to (W, Φ) .

Suppose that $\pi: Y \rightarrow X$ is a sequence of smooth blow ups of the strata of Δ . We may write

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $\pi_*\Gamma = \Delta$ and $\pi_*E = 0$.

Let $p: V \rightarrow Y$ and $q: V \rightarrow W$ resolve the induced birational map $Y \dashrightarrow W$, so that the strict transform of Φ and the exceptional locus of q has global normal crossings. We may write

$$K_V + \Psi = q^*(K_W + \Phi) + F,$$

where $\Psi \geq 0$ and $F \geq 0$ have no common components, $q_*\Psi = \Phi$ and $q_*F = 0$. Note that the coefficients of Ψ belong to I .

As $q^*(K_W + \Phi)$ is nef, Ψ has no components in common with $N_\sigma(V, K_V + \Psi)$. Thus

$$\Gamma' = \Gamma - \Gamma \wedge N_\sigma(Y, K_Y + \Gamma) = p_*\Psi$$

so that the coefficients of Γ' belong to I .

Suppose that Z is a strata of (X, Δ) which is contained in $N_\sigma(X, K_X + \Delta)$. Let $\pi: Y \rightarrow X$ blow up Z and let E be the exceptional divisor. The coefficient of E in Γ is no more than the minimum coefficient of

any component of Δ containing Z . E is a component of $\Gamma - \Gamma'$, so that the coefficient of E in Γ' is strictly less than the coefficient of any component of Δ containing Z . Since I is a finite set and (X, Δ) has only finitely many strata, it is clear that after finitely many blow ups we must have $\Gamma = \Gamma'$. \square

Lemma 2.8.4. *Let (X, Δ) be a log pair and let $\pi: X \rightarrow U$ be a morphism of quasi-projective varieties.*

Then the subset $U_0 \subset U$ of points $u \in U$ such that the fibre (X_u, Δ_u) is divisorially log terminal is constructible. Further if $U_1 \subset U_0$ is open then (X, Δ) is divisorially log terminal over U_1 .

Proof. It suffices to prove that if U_0 is dense then it contains an open subset.

Let $f: Y \rightarrow X$ be a log resolution. We may write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components. Passing to an open subset of U we may assume that (Y, Γ) is log smooth over U . As Γ_u is a boundary for a dense set of points $u \in U_0$, it follows that Γ is a boundary.

Suppose that F is an exceptional divisor of log discrepancy zero with respect to (X, Δ) , that is, coefficient one in Γ . Let $Z = f(F)$ be the centre of F in X . Note that F_u has log discrepancy zero with respect to (X_u, Δ_u) , for any $u \in U_0$. As (X_u, Δ_u) is divisorially log terminal, it follows that (X_u, Δ_u) is log smooth in a neighbourhood of the generic point of Z_u . But then (X, Δ) is log smooth in a neighbourhood of the generic point of Z and so (X, Δ) is divisorially log terminal.

But then (X_u, Δ_u) is divisorially log terminal for some open subset of points $U_1 \subset U$. \square

2.9. Good minimal models.

Lemma 2.9.1. *Let (X, Δ) be a divisorially log terminal pair, where X is projective and \mathbb{Q} -factorial.*

If (X, Δ) has a weak log canonical model then the following are equivalent

- (1) *every weak log canonical model of (X, Δ) is a semi-ample model,*
- (2) *(X, Δ) has a semi-ample model, and*
- (3) *(X, Δ) has a good minimal model.*

Proof. (1) implies (2) is clear.

We show that (2) implies (3). Suppose that $g: X \dashrightarrow Z$ is a semi-ample model of (X, Δ) . Let $p: W \rightarrow X$ be a log resolution of (X, Δ)

which also $q: W \longrightarrow Z$ resolves g . We may write

$$K_W + \Phi = p^*(K_X + \Delta) + E,$$

where $\Phi \geq 0$ and $E \geq 0$ have no common components, $p_*\Phi = \Delta$ and $p_*E = 0$. [14, 2.10] implies that (X, Δ) has a good minimal model if and only if (W, Φ) has a good minimal model.

Replacing (X, Δ) by (W, Φ) we may assume that g is a morphism. We run $f: X \dashrightarrow Y$ the $(K_X + \Delta)$ -MMP with scaling of an ample divisor over Z . Note that running the $(K_X + \Delta)$ -MMP over Z is the same as running the absolute $(K_X + \Delta + H)$ -MMP, where H is the pullback of a sufficiently ample divisor from Z . Note also that $N_\sigma(X, K_X + \Delta)$ and $N_\sigma(X, K_X + \Delta + H)$ have the same components. By (2) of (2.7.1) we may run the $(K_X + \Delta)$ -MMP with scaling until f contracts every component of $N_\sigma(X, K_X + \Delta)$. If $h: Y \longrightarrow Z$ is the induced birational morphism then h is small. As $h_*(K_Y + \Gamma) = g_*(K_X + \Delta)$ is semi-ample

$$K_Y + \Gamma = h^*h_*(K_Y + \Gamma),$$

is semi-ample and f is a good minimal model. Thus (2) implies (3).

Suppose that $f: X \dashrightarrow Y$ is a minimal model and $g: X \dashrightarrow Z$ is a weak log canonical model. Let $p: W \longrightarrow Y$ and $q: W \longrightarrow Z$ be a common resolution over X , $r: W \longrightarrow X$. Then we may write

$$p^*(K_Y + \Gamma) + E_1 = r^*(K_X + \Delta) = q^*(K_Z + \Phi) + E_2$$

where $\Gamma = f_*\Delta$, $\Phi = g_*\Delta$, $E_1 \geq 0$ is p -exceptional and $E_2 \geq 0$ is q -exceptional. As f is a minimal model and g is a weak log canonical model, every f -exceptional divisor is g -exceptional. Thus

$$p^*(K_Y + \Gamma) + E = q^*(K_Z + \Phi),$$

where $E = E_1 - E_2$ is q -exceptional. Negativity of contraction applied to q implies that $E \geq 0$, so that $E \geq 0$ is p -exceptional. Negativity of contraction applied to p implies that $E = 0$. But then $K_Y + \Gamma$ is semi-ample if and only if $K_Z + \Phi$ is semi-ample. Thus (3) implies (1). \square

Lemma 2.9.2. *Let (X, Δ) be a divisorially log terminal pair, where X is \mathbb{Q} -factorial and projective. Let A be an ample divisor.*

If (X, Δ) has a good minimal model then there is a constant $\epsilon > 0$ with the following properties:

- (1) *If $g_t: X \dashrightarrow Z_t$ is the log canonical model of $(X, \Delta + tA)$ then Z_t is independent of $t \in (0, \epsilon)$ and there is a morphism $Z_t \longrightarrow Z_0$.*
- (2) *If $h: X \dashrightarrow Y$ is a weak log canonical model of $(X, \Delta + tA)$ for some $t \in [0, \epsilon)$ then h is a semi-ample model of (X, Δ) .*

Proof. Suppose that we run $f_t: X \dashrightarrow W_t$ the $(K_X + \Delta)$ -MMP with scaling of A . [14, 2.9] implies that this MMP terminates with a minimal model, so that we may find $\epsilon > 0$ such that $f = f_0 = f_t: X \dashrightarrow W = W_t$ is independent of $t \in [0, \epsilon)$. Let $\Phi = f_*\Delta$ and let $B = f_*A$. If $C \subset W$ is a curve then

$$(K_W + \Phi + tB) \cdot C = 0 \quad \text{whenever} \quad (K_W + \Phi + sB) \cdot C = 0,$$

for all $t \in [0, \epsilon)$ and $s \in (0, \epsilon)$, since $K_W + \Phi + \lambda B$ is nef for all $\lambda \in (0, \epsilon)$. Let

$$Z_t = \text{Proj } R(X, K_X + \Delta + tA),$$

be the ample model. The induced contraction morphism $W \rightarrow Z_t$ contracts those curves C such that $(K_W + \Phi + tB) \cdot C = 0$ so that $Z = Z_t$ is independent of $t \in (0, \epsilon)$ and there is a contraction morphism $Z_t \rightarrow Z_0$. This is (1).

Let $h: X \dashrightarrow Y$ be a weak log canonical model of $(X, \Delta + tA)$. Then h is a semi-ample model of $(X, \Delta + tA)$ and there is an induced morphism $\psi: Y \rightarrow Z$.

Possibly replacing ϵ with a smaller number we may assume that h contracts every component of $N_\sigma(X, K_X + \Delta)$ by (2.7.1). Note that if P is a prime divisor which is not a component of $N_\sigma(X, K_X + \Delta)$ then $(K_X + \Delta + tA)|_P$ is big. Thus h also contracts precisely the components of $N_\sigma(X, K_X + \Delta)$. It follows that ψ is a small morphism.

If $\Gamma = h_*\Delta$, $B = h_*A$, $\Psi = \psi_*\Gamma$ and $C = \psi_*B$ then

$$K_Y + \Gamma + sB = \psi^*(K_Z + \Psi + sC),$$

for any s . By assumption $K_Z + \Psi + sC$ is ample for $s \in (0, \epsilon)$ and so $K_Y + \Gamma + sB$ is nef for $s \in (0, \epsilon)$. Thus $K_Y + \Gamma$ is nef and so h is a semi-ample model of (X, Δ) by (2.9.1). \square

Lemma 2.9.3. *Let k be any field of characteristic zero. Let (X, Δ) be a divisorially log terminal pair, where X is \mathbb{Q} -factorial and projective. Let $(\bar{X}, \bar{\Delta})$ be the corresponding pair over the algebraic closure \bar{k} of k .*

Then (X, Δ) has a good minimal model if and only if $(\bar{X}, \bar{\Delta})$ has a good minimal model.

Proof. If W is a scheme over k then \bar{W} denotes the corresponding scheme over \bar{k} . One direction is clear; if $f: X \dashrightarrow Y$ is a good minimal model of (X, Δ) then $\bar{f}: \bar{X} \dashrightarrow \bar{Y}$ is a semi-ample model of $(\bar{X}, \bar{\Delta})$ and so $(\bar{X}, \bar{\Delta})$ has a good minimal model by (2.9.1).

Conversely suppose that $(\bar{X}, \bar{\Delta})$ has a good minimal model. Pick an ample divisor A on X . We run $f: X \dashrightarrow Y$ the $(K_X + \Delta)$ -MMP with scaling of A . Then f is a weak log canonical model of $(X, \Delta + tA)$ and so $\bar{f}: \bar{X} \dashrightarrow \bar{Y}$ is a weak log canonical model of $(\bar{X}, \bar{\Delta} + t\bar{A})$. (2.9.2)

implies that we may find $\epsilon > 0$ such that \bar{f} is a semi-ample model of $(\bar{X}, \bar{\Delta})$ for $t \in [0, \epsilon)$. If $\Gamma = f_*\Delta$ then $K_{\bar{Y}} + \bar{\Gamma}$ is semi-ample so that $K_Y + \Gamma$ is semi-ample. But then f is a good minimal model of (X, Δ) . \square

3. THE MMP IN FAMILIES I

Lemma 3.1. *Let (X, Δ) be a divisorially log terminal pair and let $\pi: X \rightarrow U$ be a projective morphism, where U is smooth, affine, of dimension k and X is \mathbb{Q} -factorial. Let $0 \in U$ be a closed point such that*

- (1) *there are k divisors D_1, D_2, \dots, D_k containing 0 such that if $H_i = \pi^*D_i$ and $H = H_1 + H_2 + \dots + H_k$ is the sum then $(X, H + \Delta)$ is divisorially log terminal,*
- (2) *X_0 is reduced, $\dim X_0 = \dim X - \dim U$ and $\dim V_0 = \dim V - \dim U$, for all non canonical centres V of (X, Δ) , and*
- (3) *$\mathbf{B}_-(X_0, K_{X_0} + \Delta_0)$ contains no non canonical centres of (X_0, Δ_0) .*

Let $f: X \dashrightarrow Y$ be a step of the $(K_X + \Delta)$ -MMP. If f is birational and V is a non canonical centre of (X, Δ) then V is not contained in the indeterminacy locus of f , V_0 is not contained in the indeterminacy locus of f_0 and the induced maps $\phi: V \dashrightarrow W$ and $\phi_0: V_0 \dashrightarrow W_0$ are birational, where $W = f(V)$. Let $\Gamma = f_\Delta$. Further*

- (1) *if G_i is the pullback of D_i to Y and $G = G_1 + G_2 + \dots + G_k$ is the sum then $(Y, G + \Gamma)$ is divisorially log terminal,*
- (2) *Y_0 is reduced, $\dim Y_0 = \dim Y - \dim U$ and $\dim W_0 = \dim W - \dim U$, for all non canonical centres W of (Y, Γ) , and*
- (3) *$\mathbf{B}_-(Y_0, K_{Y_0} + \Gamma_0)$ contains no non canonical centres of (Y_0, Γ_0) .*

If V is a non kawamata log terminal centre, or $V = X$ then $\phi: V \dashrightarrow W$ and $\phi_0: V_0 \dashrightarrow W_0$ are birational contractions.

On the other hand, if f is a Mori fibre space then f_0 is not birational.

Proof. Suppose that f is birational.

As f is a step of the $(K_X + \Delta)$ -MMP and H is pulled back from U , it follows that it is also a step of the $(K_X + H + \Delta)$ -MMP, and so $(Y, G + \Gamma)$ is divisorially log terminal. As every component of Y_0 is a non kawamata log terminal centre of (Y, G) and X_0 is irreducible, it follows that Y_0 is irreducible.

Let V be a non canonical centre of (X, Δ) . Then V is a non canonical centre of $(X, H + \Delta)$. Let $g: X \rightarrow Z$ be the contraction of the extremal ray associated to f (so that $f = g$ unless f is a flip). Let $Q = g(V)$ and let $\psi: V \rightarrow Q$ be the induced morphism. As V_0 is a non canonical centre of (X_0, Δ_0) it is not contained in $\mathbf{B}_-(X_0, K_{X_0} + \Delta_0)$

and so the induced morphism $\psi_0: V_0 \rightarrow Q_0$ is birational. As Q is irreducible and dominates U , and the dimension of the fibres of $V \rightarrow Q$ are upper-semicontinuous, ψ is birational. Thus V does not belong to the indeterminacy locus of f , V_0 does not belong to the indeterminacy locus of f_0 , and both $\phi: V \dashrightarrow W$ and $\phi_0: V_0 \dashrightarrow W_0$ are birational.

Now suppose that V is a non kawamata log terminal centre or $V = X$. If V is a non kawamata log terminal centre then V is a non canonical centre and so $\phi: V \dashrightarrow W$ and $\phi_0: V_0 \dashrightarrow W_0$ are both birational. We can define divisors Σ_0 and Θ_0 on V_0 and W_0 by adjunction:

$$(K_{X_0} + \Delta_0)|_{V_0} = K_{V_0} + \Sigma_0. \quad \text{and} \quad (K_{Y_0} + \Gamma_0)|_{W_0} = K_{W_0} + \Theta_0.$$

If P is a divisor on W_0 and f is not an isomorphism at the generic point of the centre N of P on V_0 then

$$a(P; V_0, \Sigma_0) < a(P; W_0, \Theta_0) \leq 1.$$

Thus N is a non-canonical centre of (X, Δ) . Therefore N is birational to P so that N is a divisor on V_0 . Thus $\phi_0: V_0 \dashrightarrow W_0$ is a birational contraction. In particular $f_0: X_0 \dashrightarrow Y_0$ is a birational contraction and so (1–3) clearly hold. As $\phi_0: V_0 \dashrightarrow W_0$ is a birational contraction it follows that $\phi: V \dashrightarrow W$ is a birational contraction.

Suppose that f is a Mori fibre space. As the dimension of the fibres of $f: X \rightarrow Y$ are upper-semicontinuous, f_0 is not birational. \square

Lemma 3.2. *Let (X, Δ) be a divisorially log terminal pair and let $\pi: X \rightarrow U$ be a projective morphism, where U is smooth and affine and X is \mathbb{Q} -factorial. Let $\eta \in U$ be the generic point and let $0 \in U$ be a closed point. Suppose that either*

- (1) *there are k divisors D_1, D_2, \dots, D_k containing 0 such that if $H_i = \pi^* D_i$ and $H = H_1 + H_2 + \dots + H_k$ is the sum then $(X, H + \Delta)$ is divisorially log terminal,*
- (2) *X_0 is reduced, $\dim X_0 = \dim X - \dim U$ and $\dim V_0 = \dim V - \dim U$, for all non canonical centres V of (X, Δ) , and*
- (3) *$\mathbf{B}_-(X_0, K_{X_0} + \Delta_0)$ contains no non canonical centres of (X_0, Δ_0) .*

or (X, Δ) is log smooth over U and (3) holds.

If (X_0, Δ_0) has a good minimal model then we may run $f: X \dashrightarrow Y$ the $(K_X + \Delta)$ -MMP until $f_\eta: X_\eta \dashrightarrow Y_\eta$ is a minimal model of (X_η, Δ_η) and $f_0: X_0 \dashrightarrow Y_0$ is a semi-ample model of (X_0, Δ_0) . If D is a component of $\lfloor \Delta \rfloor$, E is the image of D and $\phi: D \dashrightarrow E$ is the restriction of f to D then the induced map $\phi_0: D_0 \dashrightarrow E_0$ is a semi-ample model of (D_0, Σ_0) , where Σ_0 is defined by adjunction

$$(K_{X_0} + \Delta_0)|_{D_0} = K_{D_0} + \Sigma_0.$$

Further $\mathbf{B}_-(X, K_X + \Delta)$ contains no non-canonical centres of (X, Δ) .

Proof. Suppose that (X, Δ) is log smooth over U . If D_1, D_2, \dots, D_k are k general divisors containing 0 then $(X, H + \Delta)$ is log smooth, so that (1) and (2) hold. Thus we may assume (1–3) hold.

We run $f: X \dashrightarrow Y$ the $(K_X + \Delta)$ -MMP with scaling of an ample divisor A . Let $\Gamma = f_*\Delta$ and $B = f_*A$. By construction $K_Y + tB + \Gamma$ is nef for some $t > 0$. Since $\pi: X \rightarrow U$ satisfies the hypotheses of (3.1), $f_0: X_0 \dashrightarrow Y_0$ is a weak log canonical model of $(X_0, tA_0 + \Delta_0)$.

If $K_X + \Delta$ is not pseudo-effective then this MMP ends with a Mori fibre space for some $t > 0$ and so Y_0 is covered by curves on which $K_{Y_0} + tB_0 + \Gamma_0$ is negative by (3.1). This contradicts the fact that $K_{X_0} + tA_0 + \Delta_0$ is big. Thus $K_X + \Delta$ is pseudo-effective and given any $\epsilon > 0$ we may run the MMP until $t < \epsilon$.

Since $K_{X_0} + \Delta_0$ has a good minimal model (2.9.2) implies that there is a constant $\epsilon > 0$ such that if $t \in (0, \epsilon)$ then any more steps of this MMP are an isomorphism in a neighbourhood of Y_0 . It follows that $K_{Y_\eta} + tB_\eta + \Gamma_\eta$ is nef for all $t \in (0, \epsilon)$, so that $K_{Y_\eta} + \Gamma_\eta$ is nef. Thus $f_\eta: X_\eta \dashrightarrow Y_\eta$ is a minimal model of (X_η, Δ_η) .

Suppose that D is a component of $\lfloor \Delta \rfloor$. (3.1) implies that the induced map $\phi_0: D_0 \dashrightarrow E_0$ is a birational contraction so that ϕ_0 is a semi-ample model of (D_0, Σ_0) .

As

$$(K_Y + \Gamma)|_{Y_0} = K_{Y_0} + \Gamma_0$$

is nef, it follows that $\mathbf{B}_-(X, K_X + \Delta)|_{X_0}$ is contained in the indeterminacy locus of $f_0: X_0 \dashrightarrow Y_0$. Thus $\mathbf{B}_-(X, K_X + \Delta)$ contains no non-canonical centres of (X, Δ) . \square

4. INVARIANCE OF PLURIGENERA

We will need the following result of B. Berndtsson and M. Păun.

Theorem 4.1. *Let $f: X \rightarrow \mathbb{D}$ be a projective morphism to the unit disk \mathbb{D} and let (X, Δ) be a log pair.*

If

- (1) (X, Δ) is log smooth over \mathbb{D} and $\lfloor \Delta \rfloor = 0$,
- (2) the components of Δ do not intersect,
- (3) $K_X + \Delta$ is pseudo-effective, and
- (4) $\mathbf{B}_-(X, K_X + \Delta)$ does not contain any components of Δ_0 ,

then

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \rightarrow H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0)))$$

is surjective for any integer m such that $m\Delta$ is integral.

Proof. We check that the hypotheses of [5, Theorem 0.2] are satisfied and we will use the notation established there.

We take $\alpha = 0$ and $p = m$ so that if $L = \mathcal{O}_X(m\Delta)$ then

$$p([\Delta] + \alpha) = m[\Delta] \in c_1(L),$$

is automatic. $K_X + \Delta$ is pseudo-effective by assumption. As we are assuming (4), $\nu_{\min}(\{K_X + \Delta\}, X_0) = 0$ and $\rho_{\min, \infty}^j = 0$. In particular $J = J'$ and $\Xi = 0$. As we are assuming that the components of Δ do not intersect the transversality hypothesis is automatically satisfied.

If

$$u \in H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0)))$$

is a non-zero section then we choose $h_0 = e^{-\varphi_0}$ such that $\varphi_0 \leq 0 = \varphi_\Xi$ and

$$\Theta_{h_0}(K_{X_0} + \Delta_0) = \frac{1}{m}[Z_u].$$

Since u has no poles and $\lfloor \Delta \rfloor = 0$, we have

$$\int_{X_0} e^{\varphi_0 - \frac{1}{m}\varphi_{m\Delta}} < \infty.$$

Condition (\star) is automatically satisfied, as $\rho_{\min, \infty}^j = 0$ and $J = J'$.

[5, Theorem 0.2] implies that we can extend u to

$$U \in H^0(X, \mathcal{O}_X(m(K_X + \Delta))). \quad \square$$

Theorem 4.2. *Let $\pi: X \rightarrow U$ be a projective morphism to a smooth variety U and let (X, Δ) be a log smooth pair over U .*

If $\lfloor \Delta \rfloor = 0$ then

$$h^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u))),$$

is independent of the point $u \in U$, for all positive integers m .

In particular $\kappa(X_u, K_{X_u} + \Delta_u)$ is independent of $u \in U$ and

$$f_*\mathcal{O}_X(m(K_X + \Delta)) \rightarrow H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u)))$$

is surjective for all positive integers $m > 0$ and for all $u \in U$.

Proof. Fix a positive integer m . We may assume that U is affine.

Replacing Δ by

$$\Delta_m = \frac{\lfloor m\Delta \rfloor}{m}$$

we may assume that $m\Delta$ is integral.

By (2.8.1) there is a composition of smooth blow ups of the strata of Δ such that if we write

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $\pi_*\Gamma = \Delta$ and $\pi_*E = 0$, then no two components of Γ intersect. Then (Y, Γ) is log smooth over U , $m\Gamma$ is integral and $[\Gamma] = 0$.

As

$$h^0(Y_u, \mathcal{O}_{Y_u}(m(K_{Y_u} + \Gamma_u))) = h^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u))),$$

replacing (X, Δ) by (Y, Γ) we may assume that no two components of Δ intersect.

We may assume that

$$h^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u))) \neq 0,$$

for some $u \in U$. Let F be the fixed divisor of the linear system $|m(K_{X_u} + \Delta_u)|$ and let

$$\Theta_u = \Delta_u - \Delta_u \wedge F/m.$$

There is a unique divisor $0 \leq \Theta \leq \Delta$ such that

$$\Theta|_{X_u} = \Theta_u.$$

Note that $m\Theta$ is integral,

$$f_*\mathcal{O}_X(m(K_X + \Theta)) \subset f_*\mathcal{O}_X(m(K_X + \Delta))$$

and

$$H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Theta_u))) = H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u))).$$

Replacing (X, Δ) by (X, Θ) we may assume that no component of Δ_u is in the base locus of $|m(K_{X_u} + \Delta_u)|$. In particular $\mathbf{B}_-(X_u, K_{X_u} + \Delta_u)$ does not contain any components of Δ_u . (3.2) implies that $\mathbf{B}_-(X, K_X + \Delta)$ does not contain any components of Δ_u and we may apply (4.1). \square

5. THE MMP IN FAMILIES II

Lemma 5.1. *Let (X, Δ) be a log canonical pair and let (X, Φ) be a divisorially log terminal pair, where X is \mathbb{Q} -factorial of dimension n . Let*

$$\Delta(t) = (1 - t)\Delta + t\Phi.$$

Suppose that $X \rightarrow U$ is projective. Let $f: X \dashrightarrow Y$ be a step of the $(K_X + \Delta(t))$ -MMP over U and let $\Gamma = f_\Delta$.*

Suppose $0 \in U$ is a closed point such that $K_{X_0} + \Delta_0$ is nef and (X_0, Δ_0) is log canonical. Let r be a positive integer, such that $r(K_{X_0} + \Delta_0)$ is Cartier.

If

$$0 < t \leq \frac{1}{1 + 2nr}$$

then f is $(K_X + \Delta)$ -trivial in a neighbourhood of X_0 . In particular (Y_0, Γ_0) is log canonical, $K_{Y_0} + \Gamma_0$ is nef, $r(K_{Y_0} + \Gamma_0)$ is Cartier and (Y, Γ) is log canonical in a neighbourhood of Y_0 .

Proof. Let R be the extremal ray corresponding to f .

If f is an isomorphism in a neighbourhood of X_0 there is nothing to prove and if $(K_X + \Delta) \cdot R = 0$, the result follows by [24, 3.17].

Otherwise, as $K_{X_0} + \Delta_0$ is nef, $(K_X + \Delta) \cdot R > 0$ and so $(K_X + \Phi) \cdot R < 0$. [16] implies that R is spanned by a rational curve C contained in X_0 such that

$$-(K_X + \Phi) \cdot C \leq 2n.$$

As $r(K_{X_0} + \Delta_0)$ is Cartier

$$(K_X + \Delta) \cdot C = (K_{X_0} + \Delta_0) \cdot C \geq \frac{1}{r}.$$

Thus

$$\begin{aligned} 0 &> (K_X + \Delta(t)) \cdot C \\ &= (1-t)(K_X + \Delta) \cdot C + t(K_X + \Phi) \cdot C \\ &\geq \frac{(1-t)}{r} - 2nt \\ &= \frac{1}{r} - t \frac{(1+2nr)}{r} \\ &\geq 0, \end{aligned}$$

a contradiction. \square

Lemma 5.2. *Let $(X, \Delta = S + B)$ be a divisorially log terminal pair, where $S \leq \lfloor \Delta \rfloor$ and X is \mathbb{Q} -factorial. Let $\pi: X \rightarrow U$ be a projective morphism, where U is smooth and affine. Let $0 \in U$ be a closed point, let n be the dimension of X and let r be a positive integer such that $r(K_{X_0} + \Delta_0)$ is Cartier. Fix*

$$\epsilon < \frac{1}{2nr+1}.$$

*If (X_0, Δ_0) is log canonical, $K_{X_0} + \Delta_0$ is nef but $K_X + (1-\epsilon)S + B$ is not pseudo-effective, then we may run $f: X \dashrightarrow Y$ the $(K_X + (1-\epsilon)S + B)$ -MMP over U , the steps of which are all $(K_X + \Delta)$ -trivial in a neighbourhood of X_0 , until we arrive at a Mori fibre space $\psi: Y \rightarrow Z$ such that the strict transform of S dominates Z and $K_Y + \Gamma \sim_{\mathbb{Q}} \psi^*L$, for some divisor L on Z .*

Proof. We run $f: X \dashrightarrow Y$ the $(K_X + (1-\epsilon)S + B)$ -MMP with scaling of an ample divisor over U . (5.1) implies that every step of this MMP is $(K_X + \Delta)$ -trivial in a neighbourhood of X_0 . As $K_X + (1-\epsilon)S + B$ is not

pseudo-effective this MMP ends with a Mori fibre space $\psi: Y \rightarrow Z$. As every step of this MMP is $(K_X + \Delta)$ -trivial in a neighbourhood of X_0 , it follows that the strict transform of S dominates Z . \square

Lemma 5.3. *Let (X, Δ) be a divisorially log terminal pair, where X is \mathbb{Q} -factorial and projective and Δ is a \mathbb{Q} -divisor.*

If Φ is a \mathbb{Q} -divisor such that

$$0 \leq \Delta - \Phi \leq N_\sigma(X, K_X + \Delta),$$

then (X, Φ) has a good minimal model if and only if (X, Δ) has a good minimal model.

Proof. Suppose that $f: X \dashrightarrow Y$ is a minimal model of (X, Δ) . Let $\Gamma = f_*\Delta$. (2) of (2.7.1) implies that f contracts every component of $N_\sigma(X, K_X + \Delta)$ so that

$$f_*(K_X + \Delta) = K_Y + \Gamma = f_*(K_X + \Phi).$$

Let $p: W \rightarrow X$ and $q: W \rightarrow Y$ resolve f . If we write

$$p^*(K_X + \Delta) = q^*(K_Y + \Gamma) + E,$$

then $E \geq 0$ is q -exceptional and $p_*E = N_\sigma(X, K_X + \Delta)$. It follows that if we write

$$p^*(K_X + \Phi) = q^*(K_Y + \Gamma) + F,$$

then

$$F = E - p^*(\Delta - \Phi) \geq E - p^*(N_\sigma(X, K_X + \Delta)) = E - p_*p_*E,$$

is p -exceptional. Therefore $F \geq 0$ by negativity of contraction and so f is a weak log canonical model of (X, Φ) . If f is a good minimal model of (X, Δ) then f is a semi-ample model of (X, Φ) and so (X, Φ) has a good minimal model by (2.9.1).

Now suppose that (X, Φ) has a good minimal model. We may run the $(K_X + \Phi)$ -MMP until we get a minimal model $f: X \dashrightarrow Y$ of (X, Φ) . Let $Y \rightarrow Z$ be the ample model of $K_X + \Phi$.

If $t > 0$ is sufficiently small then f is also a run of the $(K_X + \Delta_t)$ -MMP, where

$$\Delta_t = \Phi + t(\Delta - \Phi).$$

Let n be the dimension of X and let r be a positive integer such that $r(K_X + \Phi)$ is Cartier. If

$$0 < t < \frac{1}{1 + 2nr}$$

and we continue to run the $(K_X + \Delta_t)$ -MMP with scaling of an ample divisor then (5.1) implies that every step of this MMP is $(K_X + \Phi)$ -trivial, so that every step is over Z . After finitely many steps (2.7.1)

implies that we obtain a model $g: X \dashrightarrow W$ which contracts the components of $N_\sigma(X, K_X + \Delta_t)$. As the support of $N_\sigma(X, K_X + \Delta)$ is the same as the support of $N_\sigma(X, K_X + \Delta_t)$ and the support of $\Delta - \Phi$ is contained in $N_\sigma(X, K_X + \Delta)$ it follows that

$$g_*(K_X + \Delta) = g_*(K_Y + \Phi).$$

Thus $g_*(K_X + \Delta)$ is semi-ample. On the other hand g only contracts divisors in $N_\sigma(X, K_X + \Delta)$ so that (2.7.2) implies that g is a minimal model of (X, Δ) . Thus $g: X \dashrightarrow W$ is a good minimal model of (X, Δ) . \square

6. ABUNDANCE IN FAMILIES

Lemma 6.1. *Suppose that (X, Δ) is a log pair where the coefficients of Δ belong to $(0, 1] \cap \mathbb{Q}$. Let $\pi: X \rightarrow U$ be a projective morphism to a smooth affine variety U . Suppose that (X, Δ) is log smooth over U .*

If there is a closed point $0 \in U$ such that the fibre (X_0, Δ_0) has a good minimal model then the generic fibre (X_η, Δ_η) has a good minimal model.

Proof. By (2.9.3) it is enough to prove that the geometric generic fibre has a good minimal model. Replacing U by a finite cover we may therefore assume that the strata of Δ have irreducible fibres over U .

Let $f_0: Y_0 \rightarrow X_0$ be the birational morphism given by (2.8.3). As (X, Δ) is log smooth over U , the strata of Δ have irreducible fibres over U and f_0 blows up strata of Δ_0 , we may extend f_0 to a birational morphism $f: Y \rightarrow X$ which is a composition of smooth blow ups of strata of Δ . We may write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $f_*\Gamma = \Delta$ and $f_*E = 0$. (Y, Γ) is log smooth and the fibres of the components of Γ are irreducible. [14, 2.10] implies that (Y_0, Γ_0) has a good minimal model, as (X_0, Δ_0) has a good minimal model; similarly [14, 2.10] also implies that if (Y_η, Γ_η) has a good minimal model then (X_η, Δ_η) has a good minimal model.

Replacing (X, Δ) by (Y, Γ) we may assume that if

$$\Theta_0 = \Delta_0 - \Delta_0 \wedge N_\sigma(X_0, K_{X_0} + \Delta_0)$$

then $\mathbf{B}_-(X_0, K_{X_0} + \Theta_0)$ contains no strata of Θ_0 . There is a unique divisor $0 \leq \Theta \leq \Delta$ such that $\Theta|_{X_0} = \Theta_0$. (2.3.2) implies that

$$\Theta = \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta)$$

so that

$$\Delta - \Theta \leq N_\sigma(X, K_X + \Delta).$$

Hence by (5.3) and (2.9.3) it suffices to prove that (X_η, Θ_η) has a good minimal model. Replacing (X, Δ) by (X, Θ) we may assume that $\mathbf{B}_-(X_0, K_{X_0} + \Delta_0)$ contains no strata of Δ_0 . (3.2) implies that we can run $f: X \dashrightarrow Y$ the $(K_X + \Delta)$ -MMP over U to obtain a minimal model of the generic fibre. Let $\Gamma = f_*\Delta$.

Pick a component D of $\lfloor \Delta \rfloor$. Let $\phi: D \dashrightarrow E$ be the restriction of f to D . (3.2) implies that ϕ_0 is a semi-ample model of $(D_0, (\Delta_0 - D_0)|_{D_0})$. (2.9.1) implies that $(D_0, (\Delta_0 - D_0)|_{D_0})$ has a good minimal model. By induction on the dimension $(D_\eta, (\Delta_\eta - D_\eta)|_{D_\eta})$ has a good minimal model. But then $\phi_\eta: D_\eta \dashrightarrow E_\eta$ is a semi-ample model of $(D_\eta, (\Delta_\eta - D_\eta)|_{D_\eta})$.

Let $S = \lfloor \Delta \rfloor$ and $B = \{\Delta\} = \Delta - S$. Let $T = f_*S$ and $C = f_*B$. Suppose that $K_{Y_0} + (1 - \epsilon)T_0 + C_0$ is not pseudo-effective for any $\epsilon > 0$. Then $K_{X_0} + (1 - \epsilon)S_0 + B_0$ is not pseudo-effective for any $\epsilon > 0$. (4.2) implies that $K_X + (1 - \epsilon)S + B$ is not pseudo-effective for any $\epsilon > 0$. But then $K_Y + (1 - \epsilon)T + C$ is not pseudo-effective for any $\epsilon > 0$. (5.2) implies that we may run the $(K_Y + (1 - \epsilon)T + C)$ -MMP until we get to a Mori fibre space $g: Y \dashrightarrow W$, $\psi: W \rightarrow V$ over U . By assumption $g_*(K_Y + \Gamma) \sim_{\mathbb{Q}} \psi^*L$ for some divisor L .

Pick a component D of S whose image F in W dominates V . Let E be the image of D in Y . As we already observed, $\phi_\eta: D_\eta \dashrightarrow E_\eta$ is a semi-ample model of $(D_\eta, (\Delta_\eta - D_\eta)|_{D_\eta})$. As the birational map $g_0: Y_0 \dashrightarrow W_0$ is $(K_{Y_0} + \Gamma_0)$ -trivial, the birational map $g_\eta: Y_\eta \dashrightarrow W_\eta$ is also $(K_{Y_\eta} + \Gamma_\eta)$ -trivial. Then L_η is semi-ample as $(\psi^*L)|_{F_\eta}$ is semi-ample. The composition $X_\eta \dashrightarrow W_\eta$ is a semi-ample model of (X_η, Δ_η) and so (X_η, Δ_η) has a good minimal model by (2.9.1).

Otherwise, $K_{Y_0} + (1 - \epsilon)T_0 + C_0$ is pseudo-effective for some $\epsilon > 0$. If $Y_0 \rightarrow Z_0$ is the log canonical model of (Y_0, Γ_0) then T_0 does not dominate Z_0 and so if ϵ is sufficiently small then $K_{X_0} + (1 - \epsilon)S_0 + B_0$ has the same Kodaira dimension as $K_{X_0} + \Delta_0$.

$$\begin{aligned} \kappa(X_\eta, K_{X_\eta} + \Delta_\eta) &\geq \kappa(X_\eta, K_{X_\eta} + (1 - \epsilon)S_\eta + B_\eta) \\ &= \kappa(X_0, K_{X_0} + (1 - \epsilon)S_0 + B_0) \\ &= \kappa(X_0, K_{X_0} + \Delta_0) \\ &= \kappa_\sigma(X_0, K_{X_0} + \Delta_0) \\ &= \nu(Y_0, K_{Y_0} + \Gamma_0) \\ &= \nu(Y_\eta, K_{Y_\eta} + \Gamma_\eta). \end{aligned}$$

The first inequality holds as $S_\eta \geq 0$, the second equality holds by (4.2) (note that $(X_0, (1-\epsilon)S_0+B_0)$ is kawamata log terminal as (X_0, Δ_0) is divisorially log terminal) and the last equality holds as intersection numbers are deformation invariant.

We have already seen that if E is a component of T then $(K_Y + \Gamma)|_{E_\eta}$ is semi-ample. (2.5.1) implies that $(K_Y + \Gamma)|_{T_\eta}$ is semi-ample. Let $H = K_{Y_\eta} + \Gamma_\eta$. Then $H|_{T_\eta}$ is semi-ample and $aH - (K_{Y_\eta} + \Gamma_\eta)$ is nef and abundant for all $a > 1$. Thus $f_\eta: X_\eta \dashrightarrow Y_\eta$ is a good minimal model by (2.6.1). \square

Lemma 6.2. *Suppose that (X, Δ) is a log pair where the coefficients of Δ belong to $(0, 1] \cap \mathbb{Q}$. Let $\pi: X \rightarrow U$ be a projective morphism to a smooth affine variety U . Suppose that (X, Δ) is log smooth over U .*

If (X, Δ) has a good minimal model then every fibre (X_u, Δ_u) has a good minimal model.

Proof. Let $f: Y \rightarrow X$ be the birational morphism given by (2.8.3). We may write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $f_*\Gamma = \Delta$ and $f_*E = 0$. (Y, Γ) is log smooth. [14, 2.10] implies that (Y, Γ) has a good minimal model, as (X, Δ) has a good minimal model; similarly [14, 2.10] also implies that if (Y_u, Γ_u) has a good minimal model then (X_u, Δ_u) has a good minimal model.

Replacing (X, Δ) by (Y, Γ) we may assume that if

$$\Theta = \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta)$$

then $\mathbf{B}_-(X, K_X + \Theta)$ contains no strata of Θ . As

$$\Delta - \Theta \leq N_\sigma(X, K_X + \Delta)$$

(5.3) implies that (X, Θ) has a good minimal model. (2.3.2) implies that

$$\Theta_u = \Delta_u - \Delta_u \wedge N_\sigma(X_u, K_{X_u} + \Delta_u)$$

so that $\mathbf{B}_-(X_u, K_{X_u} + \Theta_u)$ contains no strata of Θ_u . Hence

$$\Delta_u - \Theta_u \leq N_\sigma(X_u, K_{X_u} + \Delta_u).$$

Hence by (5.3) it suffices to prove that (X_u, Θ_u) has a good minimal model. Replacing (X, Δ) by (X, Θ) we may assume that $\mathbf{B}_-(X_u, K_{X_u} + \Delta_u)$ contains no strata of Δ_u .

Let A be an ample divisor over U . [14, 2.7] implies that the $(K_X + \Delta)$ -MMP with scaling of A terminates $\pi: X \dashrightarrow Y$ with a good minimal model for (X, Δ) over U . Since $\mathbf{B}_-(X_u, K_{X_u} + \Delta_u)$ contains no strata of

Δ_u , (3.1) implies that $\pi_u: X_u \dashrightarrow Y_u$ is a semi-ample model of (X_u, Δ_u) . (2.9.1) implies that (X_u, Δ_u) has a good minimal model. \square

Proof of (1.2). By (6.1) the generic fibre (X_η, Δ_η) has a good minimal model. Hence we may find a good minimal model of $\pi^{-1}(U_0)$ over an open subset U_0 of U . As (X, Δ) is log smooth over U , every strata of $S = \lfloor \Delta \rfloor$ intersects $\pi^{-1}(U_0)$. Therefore we may apply [14, 1.1] to conclude that (X, Δ) has a good minimal model over U . (6.2) implies that every fibre has a good minimal model. \square

Proof of (1.3). It suffices to prove that if U_0 is dense then it contains an open subset. By (2.8.4) we may assume that (X, Δ) is divisorially log terminal and every fibre (X_u, Δ_u) is divisorially log terminal.

Let $\pi: Y \rightarrow X$ be a log resolution. We may write

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components. Passing to an open subset we may assume that (Y, Γ) is log smooth over U , so that

$$K_{Y_u} + \Gamma_u = \pi^*(K_{X_u} + \Delta_u) + E_u,$$

for all $u \in U$. [14, 2.10] implies that if (Y, Γ) has a good minimal model over U then (X, Δ) has a good minimal model over U . Similarly [14, 2.10] implies that if (X_u, Δ_u) has a good minimal model then (Y_u, Γ_u) has a good minimal model.

Replacing (X, Δ) by (Y, Γ) we may assume that (X, Δ) is log smooth over U . (1.2) implies that $U_0 = U$. \square

Lemma 6.3. *Let $\pi: X \rightarrow U$ be a projective morphism to a smooth variety U and let (X, Δ) be log smooth over U . Suppose that the coefficients of Δ belong to $(0, 1] \cap \mathbb{Q}$.*

If there is a closed point $0 \in U$ such that the fibre (X_0, Δ_0) has a good minimal model then the restriction morphism

$$\pi_* \mathcal{O}_X(m(K_X + \Delta)) \rightarrow H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0)))$$

is surjective for any $m \in \mathbb{N}$ such that $m\Delta$ is integral.

Proof. (2.3.3) implies that we may assume that $m \geq 2$. Replacing U by a finite cover we may assume that the strata of Δ have irreducible fibres over U . Since the result is local we may assume that U is affine and so we want to show that the restriction map

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \rightarrow H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0)))$$

is surjective. Cutting by hyperplanes we may assume that U is a curve.

Let $f_0: Y_0 \rightarrow X_0$ be the birational morphism given by (2.8.3). As (X, Δ) is log smooth over U , the strata of Δ have irreducible fibres

over U and f_0 blows up strata of Δ_0 , we may extend f_0 to a birational morphism $f: Y \rightarrow X$ which is a composition of smooth blow ups of strata of Δ . We may write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $f_*\Gamma = \Delta$ and $f_*E = 0$. (Y, Γ) is log smooth and the fibres of the components of Γ are irreducible. Note that $m\Gamma$ is integral and the natural maps induce isomorphisms

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \simeq H^0(Y, \mathcal{O}_Y(m(K_Y + \Gamma)))$$

and

$$H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0))) \simeq H^0(Y_0, \mathcal{O}_{Y_0}(m(K_{Y_0} + \Gamma_0)))$$

Replacing (X, Δ) by (Y, Γ) we may assume that if

$$\Theta_0 = \Delta_0 - \Delta_0 \wedge N_\sigma(X_0, K_{X_0} + \Delta_0),$$

then $\mathbf{B}_-(X_0, K_{X_0} + \Theta_0)$ contains no strata of Θ_0 . There is a unique divisor $0 \leq \Theta \leq \Delta$ such that $\Theta|_{X_0} = \Theta_0$. (2.3.2) implies that

$$\Theta = \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta).$$

(6.1) implies that we may run the $(K_X + \Theta)$ -MMP over U until we get to a minimal model $f: X \dashrightarrow Y$. (3.1) implies that f is an isomorphism in a neighbourhood of the generic point of every non kawamata log terminal centre of $(X, X_0 + \Theta)$. Let $V \subset X \times Y$ be the graph. Then $V \rightarrow X$ is an isomorphism in a neighbourhood of the generic point of each non kawamata log terminal centre of $(X, X_0 + \Theta)$. We may find a log resolution $W \rightarrow V$ of the strict transform of Θ and the exceptional divisor of $V \rightarrow Y$ which is an isomorphism in a neighbourhood of the generic point of each non kawamata log terminal centre of $(X, X_0 + \Theta)$. If $p: W \rightarrow X$ and $q: W \rightarrow Y$ are the induced morphisms then we may write

$$K_W + \Phi + W_0 = p^*(K_X + X_0 + \Theta) + E,$$

where W_0 is the strict transform of X_0 , Φ is the strict transform of $[\Theta]$ and $[E] \geq 0$ as p is an isomorphism in a neighbourhood of the generic point of each non kawamata log terminal centre of $(X, X_0 + \Theta)$.

We may also write

$$p^*((m-1)(K_X + \Theta)) = q_*f_*((m-1)(K_X + \Theta)) + F.$$

Possibly shrinking U , we may assume X_0 is \mathbb{Q} -linearly equivalent to zero. If we set

$$A = p^*(m(K_X + \Theta)) + E - F, \quad L = [A] \quad \text{and} \quad C = \{-A\}$$

then

$$\begin{aligned}
L - W_0 &= p^*(m(K_X + \Theta)) + E - F + C - W_0 \\
&= p^*(K_X + \Theta) + E + p^*((m-1)(K_X + \Theta)) - F + C - W_0 \\
&\sim_{\mathbb{Q}} K_W + \Phi + C + q^*f_*((m-1)(K_X + \Theta)).
\end{aligned}$$

$(W, \Phi + C)$ is log canonical, as $(W, \Phi + C)$ is log smooth and $\Phi + C$ is a boundary. Since all non kawamata log terminal centres of $(W, \Phi + C)$ dominate U , a generalisation of Kollár's injectivity theorem (see [18], [8, 6.3] and [4, 5.4]) implies that multiplication by a local parameter

$$H^1(W, \mathcal{O}_W(L - W_0)) \longrightarrow H^1(W, \mathcal{O}_W(L))$$

is an injective morphism and so the restriction morphism

$$H^0(W, \mathcal{O}_W(L)) \longrightarrow H^0(W_0, \mathcal{O}_{W_0}(L|_{W_0}))$$

is surjective. Note that the support of $L - \lfloor q^*f_*(m(K_X + \Theta)) \rfloor$ does not contain W_0 and

$$\begin{aligned}
L - \lfloor q^*f_*(m(K_X + \Theta)) \rfloor &= \lceil A \rceil - \lfloor q^*f_*(m(K_X + \Theta)) \rfloor \\
&\geq \lceil A - q^*f_*(m(K_X + \Theta)) \rceil \\
&= \lceil E + \frac{1}{m-1}F \rceil \\
&\geq 0.
\end{aligned}$$

We also have

$$\begin{aligned}
|L| &\subset |mp^*(K_X + \Delta) + \lceil E - F \rceil| \\
&\subset |mp^*(K_X + \Delta) + \lceil E \rceil| \\
&= |m(K_X + \Delta)|.
\end{aligned}$$

Let $q_0: W_0 \longrightarrow Y_0$ be the restriction of q to W_0 . We have

$$\begin{aligned}
|m(K_{X_0} + \Delta_0)| &= |m(K_{X_0} + \Theta_0)| \\
&= |m(K_{Y_0} + f_{0*}\Theta_0)| \\
&= |q_0^*m(K_{Y_0} + f_{0*}\Theta_0)| \\
&\subset |L|_{W_0}| \\
&= |L|_{W_0} \\
&\subset |m(K_X + \Delta)|_{X_0}. \quad \square
\end{aligned}$$

Proof of (1.4). Immediate from (6.3) and (1.2). \square

7. BOUNDEDNESS OF MODULI

Lemma 7.1. *Let w be a positive real number and let $I \subset [0, 1]$ be a set which satisfies the DCC. Fix a log smooth pair (Z, B) , where Z is a projective variety. Let \mathfrak{F} be the set of all log smooth pairs (X, Δ) such that $\text{vol}(X, K_X + \Delta) = w$, the coefficients of Δ belong to I and there is a sequence of smooth blow ups $f: X \rightarrow Z$ of the strata of B such that $f_*\Delta \leq B$.*

Then there is a sequence of blow ups $Y \rightarrow Z$ of the strata of B such that:

If $(X, \Delta) \in \mathfrak{F}$ then

$$\text{vol}(Y, K_Y + \Gamma) = w$$

where Γ is the sum of the strict transform of Δ and the exceptional divisors of the induced birational map $Y \dashrightarrow X$.

Proof. We may suppose that $1 \in I$ and that I is closed. Let

$$v(Z, B) = \sup\{\text{vol}(Z, K_Z + \Phi) \mid \Phi = f_*\Delta \text{ for some } (X, \Delta) \in \mathfrak{F}\}.$$

Let \mathfrak{D} be the set of log smooth pairs (X, Δ) such that X is projective and the coefficients of Δ belong to I . If $(X, \Delta) \in \mathfrak{F}$ then $(Z, \Phi) \in \mathfrak{D}$ so that $v(Z, B) \in \bar{V}$, where

$$V = \{\text{vol}(X, K_X + \Delta) \mid (X, \Delta) \in \mathfrak{D}\}.$$

Suppose that $g: Y \rightarrow Z$ blows up the strata of B and let C be the strict transform of B plus the g -exceptional divisors. If $(X, \Delta) \in \mathfrak{F}$ and Γ is the sum of the strict transform of Δ and the exceptional divisors of $Y \dashrightarrow X$ then

$$\text{vol}(Y, K_Y + \Gamma) \leq \text{vol}(Z, K_Z + \Phi),$$

and so $v(Y, C) \leq v(Z, B)$. As V satisfies the DCC, possibly replacing Z by a higher model Y we may assume that $v = v(Z, B)$ is minimal and it suffices to prove that $v = w$.

Clearly $v \geq w$. Pick $g_j: Y_j \rightarrow Z$ which blow up the strata of B such that the natural birational map $Y_j \dashrightarrow Y_k$ is a morphism whenever $j \geq k$ and given any $g: Y \rightarrow Z$ which blows up the strata of B we may find j such that the induced birational map $h_j: Y_j \dashrightarrow Y$ is a morphism. As $v(Y_j, C_j) = v$, where C_j is the strict transform of B plus the exceptionals, we may find pairs $(X_i^j, \Delta_i^j) \in \mathfrak{F}$ such that if Γ_i^j is the strict transform of Δ_i^j plus the exceptional divisors then

$$\lim_i \text{vol}(Y_j, K_{Y_j} + \Gamma_i^j) = v.$$

Possibly passing to a subsequence we may assume that

$$\mathrm{vol}(Y_j, K_{Y_j} + \Gamma_i^j) > v - 1/j.$$

Let (X_i, Δ_i) be the diagonal sequence, so that $X_i = X_i^i$ and $\Delta_i = \Delta_i^i$. Suppose that $g: Y \rightarrow Z$ blows up the strata of B . Let Γ_i be the strict transform of Δ_i plus the exceptional divisors. Pick j_0 such that the induced birational map $h_j: Y_j \rightarrow Y$ is a morphism for all $j \geq j_0$. Then $\Gamma_i = h_{i*}\Gamma_i^i$ for all $i \geq j_0$, so that

$$\mathrm{vol}(Y, K_Y + \Gamma_i) \geq \mathrm{vol}(Y_i, K_{Y_i} + \Gamma_i^i) \geq v - 1/i$$

and so

$$v = v(Y, C) \geq \lim_i \mathrm{vol}(Y, K_Y + \Gamma_i) \geq \lim_i \mathrm{vol}(Y_i, K_{Y_i} + \Gamma_i^i) = v.$$

Thus $\lim_i \mathrm{vol}(Y, K_Y + \Gamma_i) = v$.

Let $\Phi_i = f_{i*}\Delta_i = g_*\Gamma_i$. Possibly passing to a subsequence we may assume that $\Gamma_i \leq \Gamma_{i+1}$ so that $\Phi_i \leq \Phi_{i+1}$. Let $\Phi = \lim \Phi_i$ and $\Gamma = \lim \Gamma_i$ so that $\Phi = g_*\Gamma$.

As the set

$$\left\{ \frac{r-1}{r}i \mid r \in \mathbb{N}, i \in I \right\}$$

satisfies the DCC, by [12, (1.5)] we may find $r \in \mathbb{N}$ such that $K_{X_i} + \frac{r-1}{r}\Delta_i$ is big, for all i . In this case,

$$K_{X_i} + a\Delta_i = (1 - \epsilon)(K_{X_i} + \Delta_i) + \epsilon \left(K_{X_i} + \frac{r-1}{r}\Delta_i \right),$$

where $a = 1 - \frac{\epsilon}{r}$ so that

$$\mathrm{vol}(Y, K_Y + a\Gamma) \geq \mathrm{vol}(Y, (1 - \epsilon)(K_Y + \Gamma)) = (1 - \epsilon)^n v,$$

where $\dim Z = n$, independently of the model Y .

If we fix $\epsilon > 0$ then $(Z, (1 - \epsilon)\Phi)$ is kawamata log terminal. Hence (2.8.1) implies we may pick a birational morphism $g: Y \rightarrow Z$ such that if we write

$$K_Y + \Psi = g^*(K_Z + (1 - \epsilon)\Phi) + E$$

where $\Psi \geq 0$ and $E \geq 0$ have no common components, $g_*\Psi = (1 - \epsilon)\Phi$ and $g_*E = 0$, then no two components of Ψ intersect. In particular (Y, Ψ) is terminal.

Let

$$\Sigma = \Psi \wedge (1 - \epsilon)\Gamma.$$

As Γ is the limit of Γ_i , given $0 < \eta < 1$ it follows that we may find i such that

$$(1 - \eta)\Sigma \leq \Psi \wedge (1 - \epsilon)\Gamma_i.$$

Let $\Sigma_i \leq \Delta_i$ be the strict transform of $(1 - \eta)\Sigma$ on X_i . We have

$$\begin{aligned} \text{vol}(Y, K_Y + (1 - \eta)\Sigma) &\leq \text{vol}(X_i, K_{X_i} + \Sigma_i) \\ &\leq \text{vol}(X_i, K_{X_i} + \Delta_i) = w. \end{aligned}$$

where we used the fact that (Y, Σ) is terminal, as (Y, Ψ) is terminal, for the first inequality. Taking the limit as η goes to zero, [12, (5.3.2)] implies that

$$\text{vol}(Y, K_Y + (1 - \epsilon)\Gamma) = \text{vol}(Y, K_Y + \Sigma) \leq w.$$

But we already showed that

$$\text{vol}(Y, K_Y + (1 - \epsilon)\Gamma) \geq (1 - \epsilon)^n v,$$

independently of the model Y . Taking the limit as ϵ goes to zero, we must have $v = w$. \square

Lemma 7.2. *Let w be a positive real number and let $I \subset [0, 1]$ be a set which satisfies the DCC. Let \mathfrak{F} be a set of log canonical pairs (X, Δ) such that X is projective, the coefficients of Δ belong to I and $\text{vol}(X, K_X + \Delta) = w$.*

Then there is a projective morphism $Z \rightarrow U$ and a log smooth pair (Z, B) over U such that if $(X, \Delta) \in \mathfrak{F}$ then there is a point $u \in U$ and a birational map $f_u: X \dashrightarrow Z_u$ such that

$$\text{vol}(Z_u, K_{Z_u} + \Phi) = w$$

where $\Phi \leq B_u$ is the sum of the strict transform of Δ and the exceptional divisors of f_u^{-1} .

Proof. We may assume that $1 \in I$.

By [12, 1.3] there is a constant r such that if $(X, \Delta) \in \mathfrak{F}$ then $\phi_{r(K_X + \Delta)}$ is birational. (2.3.4) and (3.1) of [11] imply that the set \mathfrak{F} is log birationally bounded.

Therefore we may find a projective morphism $\pi: Z \rightarrow U$ and a log pair (Z, B) such that if $(X, \Delta) \in \mathfrak{D}$ then there is a point $u \in U$ and a birational map $f: X \dashrightarrow Z_u$ such that the support of the strict transform of Δ plus the f^{-1} -exceptional divisor is contained in the support of B_u . By standard arguments, see for example the proof of [11, 1.9], we may assume that (Z, B) is log smooth over U and the intersection of strata of B with the fibres is irreducible.

Let 0 be a closed point of U . Let $\mathfrak{F}_0 \subset \mathfrak{F}$ be the set of log canonical pairs (X, Δ) such that there is a birational morphism $f: X \rightarrow Z_0$ and $f_*\Delta \leq B_0$. By (7.1) there is a sequence of blow ups $g: Y_0 \rightarrow Z_0$ of the strata of B_0 such that if $(X, \Delta) \in \mathfrak{F}_0$ and Γ is the strict transform

of Δ plus the exceptional divisors then

$$\text{vol}(Y_0, K_{Y_0} + \Gamma) = w.$$

Let $g: Y \rightarrow Z$ be the sequence of blow ups of the strata of B induced by g_0 . Replacing (Z, B) by (Y, C) , where C is the sum of the strict transform of B and the exceptional divisors of g , we may assume that

$$\text{vol}(Z_0, K_{Z_0} + \Psi_0) = w,$$

where $\Psi_0 = f_*\Delta$.

Suppose that $(X, \Delta) \in \mathfrak{F}$. By a standard argument, see the proof of [11, (1.9)], we may assume that (X, Δ) is log smooth and $f: X \rightarrow Z_u$ blows up the strata of B_u . Let $h: W \rightarrow Z$ blow up the corresponding strata of B so that $W_u = X$ and $h_u = f$. Let Θ be the divisor on W such that $\Theta_u = \Delta$. Then

$$\text{vol}(W_0, K_{W_0} + \Theta_0) = \text{vol}(X, K_X + \Delta) = w,$$

by deformation invariance of plurigenera, (4.2), so that $(W_0, \Theta_0) \in \mathfrak{F}_0$.

But then

$$\text{vol}(Z_0, K_{Z_0} + \Phi_0) = w,$$

where $\Phi_0 = f_*\Theta_0$. Let $\Phi = h_*\Theta$. Then Φ_u is the strict transform of Δ plus the exceptional divisors and

$$\text{vol}(Z_u, K_{Z_u} + \Phi_u) = w,$$

by deformation invariance of plurigenera, (4.2). \square

Proposition 7.3. *Fix an integer n , a constant d and a set $I \subset [0, 1]$ which satisfies the DCC.*

Then the set $\mathfrak{F}_{lc}(n, d, I)$ of all (X, Δ) such that

- (1) *X is a union of projective varieties of dimension n ,*
- (2) *(X, Δ) is log canonical,*
- (3) *the coefficients of Δ belong to I ,*
- (4) *$K_X + \Delta$ is an ample \mathbb{Q} -divisor, and*
- (5) *$(K_X + \Delta)^n = d$,*

is bounded.

Proof. If

$$X = \prod_{i=1}^k X_i,$$

and (X_i, Δ_i) is the corresponding log canonical pair then $K_{X_i} + \Delta_i$ is ample and if $d_i = (K_{X_i} + \Delta_i)^n$ then $d = \sum d_i$. (2.4.1) and (1.6) imply that there are only finitely many tuples (d_1, d_2, \dots, d_k) .

Thus it is enough to show that the set \mathfrak{F} of irreducible pairs (X, Δ) satisfying (1–5) is bounded.

By (7.2) there is a projective morphism $Z \rightarrow U$ and a log smooth pair (Z, B) over U , such that if $(X, \Delta) \in \mathfrak{F}$ then there is a closed point $u \in U$ and a birational map $f_u: Z_u \dashrightarrow X$ such that

$$\mathrm{vol}(Z_u, K_{Z_u} + \Phi) = d,$$

where $\Phi \leq B_u$ is the sum of the strict transform of Δ and the f -exceptional divisors. (2.2.2) implies that f_u is the log canonical model of (Z_u, Φ) .

On the other hand, (1.3) implies that if we replace U by a finite disjoint union of locally closed subsets then we may assume that every fibre has of π has a log canonical model. Replacing (Z, B) by the log canonical model over U , the fibres of π are the elements of \mathfrak{F} . \square

Proof of (1.1). Let \mathfrak{F} be the set of triples (X, Δ, τ) where $(X, \Delta) \in \mathfrak{F}_{\mathrm{lc}}(n, d, I)$ and $\tau: S \rightarrow S$ is an involution of the normalisation of a divisor supported on $[\Delta]$, which fixes the different of $(K_X + \Delta)|_S$. Then τ fixes the ample divisor H , the pullback of $K_X + \Delta$ to S . Note that the set of all automorphisms which fix H is an algebraic group and the set of all involutions fixing the different is a closed subset.

It is enough to prove that \mathfrak{F} is bounded. (7.3) implies that $\mathfrak{F}_{\mathrm{lc}}(n, d, I)$ is bounded and the boundedness of τ is then automatic. \square

REFERENCES

- [1] V. Alexeev, *Boundedness and K^2 for log surfaces*, International J. Math. **5** (1994), 779–810.
- [2] ———, *Moduli spaces $M_{g,n}(W)$ for surfaces*, Higher-dimensional complex varieties (Trento, 1994), de Gruyter, Berlin, 1996, pp. 1–22.
- [3] F. Ambro, *The moduli b -divisor of an lc-trivial fibration*, Compos. Math. **141** (2005), no. 2, 385–403.
- [4] ———, *An injectivity theorem*, Compos. Math. **150** (2014), no. 6, 999–1023.
- [5] B. Berndtsson and M. Păun, *Quantitative extensions of pluricanonical forms and closed positive currents*, Nagoya Math. J. **205** (2012), 25–65.
- [6] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.
- [7] O. Fujino, *Special termination and reduction to pl flips*, Flips for 3-folds and 4-folds (Alessio Corti, ed.), Oxford University Press, 2005, pp. 55–65.
- [8] ———, *Fundamental theorems for the log minimal model program*, Publ. Res. Inst. Math. Sci. **47** (2011), no. 3, 727–789.
- [9] O. Fujino and Y. Gongyo, *Log pluricanonical representations and the abundance conjecture*, Compos. Math. **150** (2014), no. 4, 593–620.
- [10] C. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type II*, J. Amer. Math. Soc. **23** (2010), no. 2, 469–490, arXiv:0808.1929.

- [11] C. Hacon, J. McKernan, and C. Xu, *On the birational automorphisms of varieties of general type*, Ann. of Math. **177** (2013), no. 3, 1077–1111.
- [12] ———, *ACC for log canonical thresholds*, Ann. of Math. **180** (2014), no. 2, 523–571.
- [13] C. Hacon and C. Xu, *On finiteness of B -representation and semi-log canonical abundance*, To appear: Minimal models and extremal rays, arXiv:1107.4149.
- [14] ———, *Existence of log canonical closures*, Invent. Math. **192** (2013), no. 1, 161–195.
- [15] Y. Kawamata, *Pluricanonical systems on minimal algebraic varieties*, Invent. math. **79** (1985), 567–588.
- [16] ———, *On the length of an extremal rational curve*, Invent. Math. **105** (1991), 609–611.
- [17] J. Kollár, *Moduli of higher dimensional varieties*, Book to appear.
- [18] ———, *Higher direct images of dualizing sheaves I*, Ann. Math. **123** (1986), 11–42.
- [19] ———, *Log surfaces of general type; some conjectures*, Contemp. Math. **162** (1994), 261–275.
- [20] ———, *Rational Curves on Algebraic Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 32, Springer, 1996.
- [21] ———, *Sources of log canonical centers*, 2011, arXiv:1107.2863v2.
- [22] ———, *Moduli of varieties of general type*, Handbook of Moduli: Volume II, Adv. Lect. Math. (ALM), vol. 24, Int. Press, Somerville, MA, 2013, arXiv:1008.0621v1, pp. 115–130.
- [23] ———, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013, With a collaboration of Sándor Kovács.
- [24] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge tracts in mathematics, vol. 134, Cambridge University Press, 1998.
- [25] J. Kollár and N. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. math. **91** (1988), 299–338.
- [26] R. Lazarsfeld and M. Mustață, *Convex bodies associated to linear series*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 5, 783–835.
- [27] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.
- [28] X. Wang and C. Xu, *Nonexistence of asymptotic GIT compactification*, Duke Math J. **163** (2014), no. 12, 2217–2241.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, 155 SOUTH 1400
EAST, JWB 233, SALT LAKE CITY, UT 84112, USA
E-mail address: hacon@math.utah.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO,
9500 GILMAN DRIVE # 0112, LA JOLLA, CA 92093-0112, USA
E-mail address: mckernan@math.ucsd.edu

BEIJING INTERNATIONAL CENTER OF MATHEMATICS RESEARCH, 5 YIHEYUAN
ROAD, HAIDIAN DISTRICT, BEIJING 100871, CHINA
E-mail address: cyxu@math.pku.edu.cn